SOME PROPERTIES OF LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

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Abstract. We introduce the concept of (I, J)-coweakly Laskerian modules and study their properties concerning to local cohomology modules with respect to a pair of ideals. We also study some properties of the local cohomology modules with support contained in $\mathrm{Max}(R)$ and the relationships on the weakly artinianness of the modules $H^i_{I,J}(M)$ and $H^i_I(M)$.

Key words: Local cohomology, weakly Laskerian, weakly artinian
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1. Introduction

Throughout the paper, R is commutative Noetherian ring with identity. Let I, J be ideals of R and M an R-module.

In 2009, Takahashi, Yoshino and Yoshizawa ([10]) introduced local cohomology modules with respect to a pair of ideals which is an extension of the local cohomology modules of Grothendieck. The submodule $\Gamma_{I,J}(M)$ of M is defined as

$$\Gamma_{I,J}(M) = \{x \in M \mid I^n x \subseteq Jx \text{ for some } n \geq 0\}.$$

The functor $\Gamma_{I,J}$ is left exact, covariant, R-linear from the category of R-modules to itself. For each $i \geq 0$, the i-th right derived functor of $\Gamma_{I,J}$ is called the i-th local cohomology functor with respect to (I, J) and denoted by $H^i_{I,J}$. Note that if $J = 0$, then $H^i_{I,J}$ coincides with Grothendieck's local cohomology functor H^i_I.

In 2014, Abbasi, H. Roshan-Shekalguorabia and Rasht in [1] introduced a new Serre class of R-modules which contains the class of weakly

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Laskerian modules and the class of I-minimax R-modules. The authors defined M to be an I-weakly Laskerian R-module if for any submodule N of M the set $\mathrm{Ass}_R(\Gamma_I(M/N))$ is finite, and give some properties of I-weakly Laskerian modules. They proved that if M is an I-weakly Laskerian R-module and t is a non-negative integer such that $H^i_I(M)$ is I-weakly Laskerian for all $i < t$, then set of associated primes of $H^t_I(M)$ is finite.

The purpose of this paper is to introduce the concept of (I, J)-coweakly Laskerian modules and study some their properties concerning to local cohomology modules with respect to a pair of ideals. Then we have a result that relates to the finiteness of the associated primes of $H^i_{I,J}(M)$. We also have some results on the weakly artinianness of the modules $H^i_{I,J}(M)$.

The organization of the paper is as follows.

In the next Section, we introduce the concept of (I, J)-*coweakly Laskerian* modules. An R-module M is called an (I, J)-*coweakly Laskerian* R-module if $\mathrm{Supp}_R(M) \subseteq W(I, J)$ and $\mathrm{Ext}^i_R(R/I, M)$ is J-weakly Laskerian for all $i \geq 0$, where

$$W(I, J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some } n \geq 0\}.$$

We will see in Theorem 2.7 that if $\mathrm{Ext}^i_R(R/I, M)$ is J-weakly Laskerian for all $i \geq 0$ and n is a non-negative integer such that $H^i_{I,J}(M)$ is (I, J)-coweakly Laskerian for all $i \neq n$, then $H^n_{I,J}(M)$ is (I, J)-coweakly Laskerian. Theorem 2.9 shows that if $H^i_{I,J}(M)$ is (I, J)-coweakly Laskerian for all $i < n$ and $\mathrm{Ext}^n_R(R/I, M)$ is J-weakly Laskerian, then $\mathrm{Hom}_R(R/I, H^n_{I,J}(M))$ is J-weakly Laskerian. As a consequence of Theorem 2.9, we obtain a finite result of the set $\mathrm{Ass}_R(H^n_{I,J}(M)/JH^n_{I,J}(M))$ (Corollary 2.11).

The last Section deals with the modules which have support contained in $\mathrm{Max}(R)$. An R-module M is weakly artinian if $\mathrm{Supp}(M)$ contains only finitely many maximal ideals (see [5]). This is an extension of artinian modules and it has a close relation to weakly Laskerian modules. Theorem 3.4 says that $H^d_{I,J}(M)/JH^d_{I,J}(M)$ is weakly artinian if M is a weakly Laskerian R-module with $d = \dim M$. Moreover, if $(R, \mathfrak{m})$ is a local ring, then $H^d_{I,J}(M)$ is artinian or $\mathrm{Supp}(H^d_{I,J}(M))$ is finite. A relationship between the weakly artinianness of $H^i_{I,J}(M)$ and $H^i_I(M)$ will be shown in Theorem 3.7. This section is closed by Theorem 3.11 which shows that if M is a weakly Laskerian R-module, then

$$\begin{aligned}
\inf\{w - f - \mathrm{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \tilde{W}(I, J)\} &= \inf\{i \mid H^i_{I,J}(M) \text{ is not weakly artinian}\} \\
&= \inf\{i \mid H^i_{I,J}(M) \ncong H^i_{\mathfrak{m},J}(M)\} \\
&= \inf\{i \mid H^i_{I,J}(M) \ncong H^i_{\mathfrak{m}}(M)\}.
\end{aligned}$$

2. On coweakly Laskerian modules

We first recall the definition and some properties of J-weakly Laskerian modules.

Definition 2.1. [1, 2.1] An R-module M is said to be J-weakly Laskerian if the set of associated primes of the J-torsion submodule of any quotient module of M is finite; i.e., for any submodule N of M, the set $\mathrm{Ass}_R(\Gamma_J(M/N))$ is finite.

Lemma 2.2. [1] *The following statements are true:*

(i) *The class of J-weakly Laskerian modules is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of all R-modules. In particular, any finite sum of J-weakly Laskerian modules is J-weakly Laskerian.*

(ii) *Let M and N be two R-modules. If M is finitely generated and N is J-weakly Laskerian, then $\mathrm{Ext}^i_R(M,N)$ and $\mathrm{Tor}^R_i(M,N)$ are J-weakly Laskerian for all $i \geq 0$.*

(iii) *If M is a J-weakly Laskerian R-module, then $\Gamma_J(M)$ is weakly Laskerian.*

Definition 2.3. An R-module M is said to be (I,J)-coweakly Laskerian if $\mathrm{Supp}(M) \subseteq W(I,J)$ and $\mathrm{Ext}^i_R(R/I,M)$ is J-weakly Laskerian for every $i \geq 0$.

The class of (I,J)-coweakly Laskerian modules is larger than the class of (I,J)-cofinite modules.

Here are some elementary properties of (I,J)-coweakly Laskerian modules.

Proposition 2.4. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence. If there are two modules which are (I,J)-coweakly Laskerian, then so is the third.*

Proof. The short exact sequence $0 \to L \to M \to N \to 0$ gives rise to a long exact squence

$$\cdots \to \mathrm{Ext}^i_R(R/I,L) \to \mathrm{Ext}^i_R(R/I,M) \to \mathrm{Ext}^i_R(R/I,N) \to \cdots .$$

Note that $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(L) \cup \mathrm{Supp}_R(N)$. The assertion follows from [1, 2.2]. □

Proposition 2.5. *Let M be an R-module and t a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-coweakly Laskerian for all $i \leq t$. Then $\mathrm{Ext}^i_R(R/I,M)$ is J-weakly Laskerian for all $i \leq t$.*

Proof. The proof is by induction on t. When $t = 0$, since $H^0_{I,J}(M)$ is (I,J)-coweakly Laskerian, $\mathrm{Hom}_R(R/I, H^0_{I,J}(M))$ is J-weakly Laskerian. The claim in this case follows from the isomorphism

$$\mathrm{Hom}_R(R/I, H^0_{I,J}(M)) \cong \mathrm{Hom}_R(R/I, M).$$

Let $t > 0$, the short exact sequence

$$0 \to \Gamma_{I,J}(M) \to M \to M/\Gamma_{I,J}(M) \to 0$$

induces a long exact sequence

$$\cdots \mathrm{Ext}^i_R(R/I, \Gamma_{I,J}(M)) \to \mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, M/\Gamma_{I,J}(M)) \cdots .$$

Since $\Gamma_{I,J}(M)$ is an (I,J)-coweakly Laskerian R-module, $\mathrm{Ext}^i_R(R/I, \Gamma_{I,J}(M))$ is J-weakly Laskerian for all $i \geq 0$. The proof is complete by showing that $\mathrm{Ext}^i_R(R/I, M/\Gamma_{I,J}(M))$ is J-weakly Laskerian for all $i \leq t$. Note that $H^i_{I,J}(M) \cong H^i_{I,J}(M/\Gamma_{I,J}(M))$ for all $i > 0$. Let $\overline{M} = M/\Gamma_{I,J}(M)$, E be an injective hull of $\overline{M}$ and $L = E/\overline{M}$. From the short exact sequence

$$0 \to \overline{M} \to E \to L \to 0$$

there are isomorphisms

$$H^i_{I,J}(L) \cong H^{i+1}_{I,J}(\overline{M}) \text{ and } \mathrm{Ext}^i_R(R/I, L) \cong \mathrm{Ext}^{i+1}_R(R/I, \overline{M})$$

for all $i \geq 0$. By the assumption, $H^i_{I,J}(L)$ is (I,J)-coweakly Laskerian for all $i \leq t-1$. It follows from the inductive hypothesis that $\mathrm{Ext}^i_R(R/I, L)$ is J-weakly Laskerian for all $i \leq t-1$. This implies that $\mathrm{Ext}^i_R(R/I, \overline{M})$ is J-weakly Laskerian for all $i \leq t$. □

Corollary 2.6. *Let M be an R-module such that $H^i_{I,J}(M)$ is (I,J)-coweakly Laskerian for all $i \geq 0$. Then $\mathrm{Ext}^i_R(R/I, M)$ is J-weakly Laskerian for all $i \geq 0$.*

We will show some conditions such that the module $H^i_{I,J}(M)$ is (I,J)-coweakly Laskerian for all $i \geq 0$. The following result is an extension of [7, 2.9].

Theorem 2.7. *Let M be an R-module such that $\mathrm{Ext}^i_R(R/I, M)$ is J-weakly Laskerian for all $i \geq 0$. If n is a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-coweakly Laskerian for all $i \neq n$, then $H^n_{I,J}(M)$ is (I,J)-coweakly Laskerian.*

Proof. We use induction on n. When $n = 0$ and $\overline{M} = M/\Gamma_{I,J}(M)$. Since $H^i_{I,J}(M) \cong H^i_{I,J}(\overline{M})$ for all $i > 0$ and $\Gamma_{I,J}(\overline{M}) = 0$, it follows from the hypothesis that $H^i_{I,J}(\overline{M})$ is (I,J)-coweakly Laskerian for all $i \geq 0$. By

2.6, $\operatorname{Ext}^i_R(R/I, \overline{M})$ is J-weakly Laskerian for all $i \geq 0$. Combining the long exact sequence

$$\cdots \operatorname{Ext}^i_R(R/I, \Gamma_{I,J}(M)) \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, \overline{M}) \cdots$$

with the assumption, we can conclude that $\operatorname{Ext}^i_R(R/I, H^0_{I,J}(M))$ is J-weakly Laskerian for all $i \geq 0$. Hence $H^0_{I,J}(M)$ is (I,J)-coweakly Laskerian.

Now let $n > 0$ and the statement is true for all $i < n$. Let E be an injective hull of $\overline{M}$ and $L = E/\overline{M}$. From the short exact sequence

$$0 \to \overline{M} \to E \to L \to 0$$

there are isomorphisms

$$H^i_{I,J}(L) \cong H^{i+1}_{I,J}(\overline{M}) \text{ and } \operatorname{Ext}^i_R(R/I, L) \cong \operatorname{Ext}^{i+1}_R(R/I, \overline{M})$$

for all $i \geq 0$. By the assumption, $H^i_{I,J}(L)$ is (I,J)-coweakly Laskerian for all $i \neq n-1$ and $\operatorname{Ext}^i_R(R/I, L)$ is J-weakly Laskerian for all $i \geq 0$. It follows from the inductive hypothesis that $H^{n-1}_{I,J}(L)$ is (I,J)-coweakly Laskerian. Therefore $H^n_{I,J}(\overline{M})$ is (I,J)-coweakly Laskerian and the proof is complete. □

Corollary 2.8. *Let I be a principal ideal of R and M a J-weakly Laskerian R-module. Then $H^i_{I,J}(M)$ is (I,J)-coweakly Laskerian for all $i \geq 0$.*

Proof. It follows from [10, 4.10] that $H^i_{I,J}(M) = 0$ for all $i > 1$. By [1, 2.3 (ii)], $\operatorname{Ext}^i_R(R/I, M)$ is J-weakly Laskerian for all $i \geq 0$. The conclusion follows from 2.7. □

Theorem 2.9. *Let M be an R-module and n a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-coweakly Laskerian for all $i < n$.*

(i) *If $\operatorname{Ext}^n_R(R/I, M)$ is J-weakly Laskerian, then $\operatorname{Hom}_R(R/I, H^n_{I,J}(M))$ is J-weakly Laskerian.*
(ii) *If $\operatorname{Ext}^{n+1}_R(R/I, M)$ is a J-weakly Laskerian R-module, then $\operatorname{Ext}^1_R(R/I, H^n_{I,J}(M))$ is J-weakly Laskerian.*
(iii) *If $\operatorname{Ext}^{n+1}_R(R/I, M)$ and $\operatorname{Ext}^{n+2}_R(R/I, M)$ are J-weakly Laskerian, then $\operatorname{Hom}_R(R/I, H^{n+1}_{I,J}(M))$ is J-weakly Laskerian if and only if $\operatorname{Ext}^2_R(R/I, H^n_{I,J}(M))$ is J-weakly Laskerian.*

Proof. Let $F = \operatorname{Hom}_R(R/I, -)$ and $G = \Gamma_{I,J}(-)$ be functors from the category of R-modules to itself. For each R-module M, we have $FG(M) = \operatorname{Hom}_R(R/I, M)$. If E is an injective R-module, then $R^iF(GE) = 0$ for all $i > 0$. By [8, 10.47] we have a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_R(R/I, H^q_{I,J}(M)) \underset{p}{\Rightarrow} \operatorname{Ext}^{p+q}_R(R/I, M).$$

By the assumption, $E_2^{p,q}$ is J-weakly Laskerian for all $p \geq 0, q < n$. Let t be a non-negative integer, there is a filtration Φ of submodules of $H^t = \mathrm{Ext}_R^t(R/I, M)$

$$0 = \Phi^{t+1}H^t \subseteq \Phi^t H^t \subseteq \cdots \subseteq \Phi^1 H^t \subseteq \Phi^0 H^t = H^t$$

such that

$$E_\infty^{i,t-i} \cong \Phi^i H^t / \Phi^{i+1} H^t$$

for all $0 \leq i \leq t$.

(i) Since $H^n = \mathrm{Ext}_R^n(R/I, M)$ is J-weakly Laskerian, $E_\infty^{i,n-i}$ is J-weakly Laskerian for all $0 \leq i \leq n$. On the other hand, we have

$$E_\infty^{0,n} = E_{n+1}^{0,n} = \ker(E_n^{0,n} \to E_n^{n,1}).$$

Since $E_n^{n,1}$ is a subquotient of $E_2^{n,1}$, by the hypothesis $E_n^{n,1}$ is J-weakly Laskerian. This implies that $E_n^{0,n}$ is J-weakly Laskerian. By the same arguments, we can conclude that $E_{n-1}^{0,n}, E_{n-2}^{0,n}, \ldots, E_2^{0,n}$ are J-weakly Laskerian. In particular, $\mathrm{Hom}_R(R/I, H_{I,J}^n(M)) = E_2^{0,n}$ is J-weakly Laskerian, as required.

(ii) We consider the following spectral homomorphisms

$$0 \to E_r^{1,n} \to E_r^{1+r,n-r+1}$$

for all $r \geq 2$. We see that $E_{n+2}^{1,n} = E_{n+3}^{1,n} = \ldots = E_\infty^{1,n}$ and $E_\infty^{1,n}$ is a subquotient of $\mathrm{Ext}_R^{n+1}(R/I, M)$. By the hypothesis, $E_{n+2}^{1,n}$ and $E_r^{r+1,n-r+1}$ are J-weakly Laskerian for all $r \geq 2$. On the other hand, since $E_{r+1}^{1,n} = \ker(E_r^{1,n} \to E_r^{r+1,n-r+1})$, it follows that $E_r^{1,n}$ is J-weakly Laskerian for all $r \geq 2$.

(iii) We consider the following spectral homomorphisms

$$0 \to E_2^{0,n+1} \xrightarrow{d_2^{0,n+1}} E_2^{2,n} \xrightarrow{d_2^{2,n}} E_2^{4,n-1}$$

and

$$0 \to E_r^{2,n} \xrightarrow{d_r^{2,n}} E_r^{r+2,n-r+1}$$

for all $r \geq 3$. Note that $E_{n+2}^{2,n} = E_{n+3}^{2,n} = \ldots = E_\infty^{2,n} \cong \Phi^2 H^{n+2}/\Phi^3 H^{n+2}$. Moreover, by the hypothesis $H^{n+2} = \mathrm{Ext}_R^{n+2}(R/I, M)$ and $E_r^{r+2,n-r+1}$ are J-weakly Laskerian for all $r \geq 3$. This implies that $E_3^{2,n}$ J-weakly Laskerian. We see that $E_2^{2,n}$ is J-weakly Laskerian if and only if $\ker d_2^{2,n}$ is J-weakly Laskerian. Since $E_3^{2,n} = \ker d_2^{2,n}/\mathrm{Im} d_2^{0,n+1}$, it follows that $\ker d_2^{2,n}$ is J-weakly Laskerian if and only if $\mathrm{Im} d_2^{0,n+1}$ is J-weakly Laskerian if and only if $E_2^{0,n+1}$ is J-weakly Laskerian, and the proof is complete. □

Theorem 2.10. *Let M be a J-weakly Laskerian R-module and n a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-coweakly Laskerian for all $i < n$. Assume that N is a J-weakly Laskerian submodule of $H^n_{I,J}(M)$ such that $\mathrm{Ext}^1_R(R/I, N)$ is J-weakly Laskerian. Then $\mathrm{Hom}_R(R/I, H^n_{I,J}(M)/N)$ is J-weakly Laskerian.*

Proof. The short exact sequence

$$0 \to N \to H^n_{I,J}(M) \to H^n_{I,J}(M)/N \to 0$$

induces an exact sequence

$$\mathrm{Hom}_R(R/I, H^n_{I,J}(M)) \to \mathrm{Hom}_R(R/I, H^n_{I,J}(M)/N) \to \mathrm{Ext}^1_R(R/I, N).$$

By the assumption and 2.9, it follows that $\mathrm{Hom}_R(R/I, H^n_{I,J}(M)/N)$ is J-weakly Laskerian and the proof is complete. □

We now have a result about the finiteness of the set $\mathrm{Ass}_R(H^n_{I,J}(M)/JH^n_{I,J}(M))$.

Corollary 2.11. *Let M be a J-weakly Laskerian R-module and n a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-coweakly Laskerian for all $i < n$. If $JH^n_{I,J}(M)$ is a J-weakly Laskerian R-module, then the set $\mathrm{Ass}_R(H^n_{I,J}(M)/JH^n_{I,J}(M))$ is finite.*

Proof. It follows from 2.10 that $\mathrm{Hom}_R(R/I, H^n_{I,J}(M)/JH^n_{I,J}(M))$ is weakly Laskerian. Note that $H^n_{I,J}(M)/JH^n_{I,J}(M)$ is an I-torsion R-module. Consequently,

$$\mathrm{Ass}_R(H^n_{I,J}(M)/JH^n_{I,J}(M)) = \mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^n_{I,J}(M)/JH^n_{I,J}(M)))$$

a finite set. □

3. The weakly artinianness of local cohomology modules

We begin by recalling the definition of weakly artinian modules. An R-module M is said to be weakly artinian if $E_R(M)$, its injective envelope, can be written as $E_R(M) = \bigoplus_{i=1}^n E_R(R/\mathfrak{m}_i)^{\mu_0(\mathfrak{m}_i;M)}$, where $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are maximal ideals of R. It follows from [5, 2.3 (a)] that M is weakly artinian if and only if $\mathrm{Supp}(M)$ consists of finitely many maximal ideals. All artinian modules are weakly artinian modules. The class of weakly artinian modules is a Serre subcategory of the category of R-modules. It should be mentioned that M is weakly artinian if and only if M is weakly Laskerian and $\mathrm{Supp}(M)$ consists of maximal ideals ([5, 2.3]).

An $R-$module M is called *I-torsion-free* if $\Gamma_I(M) = 0$. It is well-known that a finitely generated $R-$module M is I-torsion-free if and only if I contains a non-zerodivisor on M ([2, 2.1.1(ii)]). In the following lemma we give a similar property for weakly Laskerian $R-$modules.

Lemma 3.1. *Let I be an ideal of R and M a weakly Laskerian R-module. Then M is an I-torsion-free R-module if and only if there is an element $x \in I$ which is M-regular.*

Proof. The proof is similar to [2, 2.1.1(ii)], note that the set $\operatorname{Ass}(M)$ is finite. □

We have a result on the weakly artinianness of local cohomology modules with respect to a pair of ideals.

Proposition 3.2. *Let M be a weakly artinian R-module. Then $H^i_{I,J}(M)$ is weakly artinian for all $i \geq 0$.*

Proof. By the hypothesis, we have $\dim M = 0$ hence $H^i_{I,J}(M) = 0$ for all $i > 0$. It follows from [5, 2.3 (c)] that $H^0_{I,J}(M)$ is weakly artinian, which completes the proof. □

It is well-known that if $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module, then $H^i_{\mathfrak{m}}(M)$ is artinian for all $i \geq 0$ (see [2]). Note that, if M is an arbitrary R-module, then $H^i_{\mathfrak{m}}(M)$ is weakly artinian for all $i \geq 0$.

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a local ring and $\sqrt{I+J} = \mathfrak{m}$. Then $H^i_{I,J}(M)/JH^i_{I,J}(M)$ is weakly artinian for all $i \geq 0$.*

Proof. It follows from [10, 1.4] that $H^i_{I,J}(M) \cong H^i_{\mathfrak{m},J}(M)$ for all $i \geq 0$. Note that $\operatorname{Supp}(H^i_{\mathfrak{m},J}(M)/JH^i_{\mathfrak{m},J}(M)) \subseteq \{\mathfrak{m}\}$ and then the proof is complete by [5, 2.3 (b)]. □

Hajikarimi proved in [5, 2.11] that if M is a weakly Laskerian R-module of finite dimension d, then $H^d_I(M)$ is weakly artinian. Chu and Wang showed in [3, 2.1] that in a local ring and M is finitely generated with $\dim M = d$, the module $H^d_{I,J}(M)$ is artinian. Now we give an extension of these results in the case M is weakly Laskerian.

Theorem 3.4. *Let M be a weakly Laskerian R-module with $\dim M = d < \infty$. Then $H^d_{I,J}(M)/JH^d_{I,J}(M)$ is weakly artinian.*

Proof. We prove by induction on d. If $d = 0$, then $\operatorname{Ass}(M) \subseteq \operatorname{Max}(R)$. Moreover, $\operatorname{Ass}(M)$ is a finite set since M is weakly Laskerian. By [5, 2.3 (b)], M is weakly artinian and then $H^0_{I,J}(M)/JH^0_{I,J}(M)$ is weakly artinian.

Let $d > 0$. The short exact sequence

$$0 \to \Gamma_J(M) \to M \to M/\Gamma_J(M) \to 0$$

induces an exact sequence

$$H^d_{I,J}(\Gamma_J(M)) \to H^d_{I,J}(M) \to H^d_{I,J}(M/\Gamma_J(M)) \to 0.$$

By [10, 2.5], $H^d_{I,J}(\Gamma_J(M)) \cong H^d_I(\Gamma_J(M))$. Moreover, $\Gamma_J(M)$ is weakly Lakerian with $\dim \Gamma_J(M) \leq d$, hence $H^d_I(\Gamma_J(M))$ is weakly artinian by [5, 2.11].

Let $\overline{M} = M/\Gamma_J(M)$, it is sufficient to show that $H^d_{I,J}(\overline{M})/JH^d_{I,J}(\overline{M})$ is weakly artinian. By 3.1, there is an element $x \in J$ which is $\overline{M}$-regular. Now the short exact sequence

$$0 \to \overline{M} \xrightarrow{.x} \overline{M} \to \overline{M}/x\overline{M} \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^{d-1}_{I,J}(\overline{M}/x\overline{M}) \xrightarrow{f} H^d_{I,J}(\overline{M}) \xrightarrow{.x} H^d_{I,J}(\overline{M}) \to 0.$$

Since $\overline{M}/x\overline{M}$ is weakly Laskerian with $\dim(\overline{M}/x\overline{M}) \leq d-1$, we conclude that $H^{d-1}_{I,J}(\overline{M}/x\overline{M})/JH^{d-1}_{I,J}(\overline{M}/x\overline{M})$ is weakly artinian by the inductive hypothesis. The short exact sequence

$$0 \to \operatorname{Im} f \to H^d_{I,J}(\overline{M}) \xrightarrow{.x} H^d_{I,J}(\overline{M}) \to 0$$

induces a long exact sequence

$$\cdots \operatorname{Im} f/J \operatorname{Im} f \to H^d_{I,J}(\overline{M})/JH^d_{I,J}(\overline{M}) \xrightarrow{.x} H^d_{I,J}(\overline{M})/JH^d_{I,J}(\overline{M}) \to 0.$$

As $x \in J$, it follows that $H^d_{I,J}(\overline{M})/JH^d_{I,J}(\overline{M})$ is a homomorphic image of $\operatorname{Im} f/J \operatorname{Im} f$. But $\operatorname{Im} f/J \operatorname{Im} f$ is weakly artinian by [5, 2.3 (c)]. Hence $H^d_{I,J}(\overline{M})/JH^d_{I,J}(\overline{M})$ is weakly artinian. □

From the proof of 3.4 (i), we see that if $\dim(M/\Gamma_J(M)) < \dim M$, then $H^{\dim M}_{I,J}(M)$ is weakly artinian.

Corollary 3.5. *Let M be a weakly Laskerian R-module with $\dim M = d < \infty$. Then $\operatorname{Supp}(H^d_{I,J}(M)/JH^d_{I,J}(M))$ is finite.*

Proposition 3.6. *Let M be an R-module and $n \geq 1$ an integer such that $\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \operatorname{Max}(R)$ for all $i < n$. If $\operatorname{Ext}^1_R(R/I, JH^i_{I,J}(M))$ and $\operatorname{Ext}^i_R(R/I, M)$ are weakly Laskerian for all $i < n$, then $H^i_{I,J}(M)/JH^i_{I,J}(M)$ is weakly artinian for all $i < n$.*

Proof. We prove by induction on n. Let $n = 1$, since $\operatorname{Supp}_R(H^0_{I,J}(M)) \subseteq \operatorname{Max}(R)$ and $0 :_{\Gamma_{I,J}(M)} I$ is weakly Laskerian, by [5, 2.3(v)] $0 :_{\Gamma_{I,J}(M)} I$ is weakly artinian. The short exact sequence

$$0 \to J\Gamma_{I,J}(M) \to \Gamma_{I,J}(M) \to \Gamma_{I,J}(M)/J\Gamma_{I,J}(M) \to 0$$

induces the following exact sequence

$$\cdots \operatorname{Hom}_R(R/I, \Gamma_{I,J}(M)) \to \operatorname{Hom}_R(R/I, \Gamma_{I,J}(M)/J\Gamma_{I,J}(M))$$

$$\to \mathrm{Ext}^1_R(R/I, J\Gamma_{I,J}(M)) \to \mathrm{Ext}^1_R(R/I, \Gamma_{I,J}(M)) \to \cdots$$

Note that the support of $\mathrm{Ext}^1_R(R/I, J\Gamma_{I,J}(M))$ is contained in $\mathrm{Max}(R)$ and it is weakly Laskerian. Therefore, $\mathrm{Ext}^1_R(R/I, J\Gamma_{I,J}(M))$ is weakly artinian by [5, 2.3(v)]. This implies that $\mathrm{Hom}_R(R/I, \Gamma_{I,J}(M)/J\Gamma_{I,J}(M))$ is weakly artinian. Moreover, $\Gamma_{I,J}(M)/J\Gamma_{I,J}(M))$ is an I-torsion R-module, it follows from [5, 2.8] that $\Gamma_{I,J}(M)/J\Gamma_{I,J}(M))$ is weakly artinian.

Let $n > 1$ and assume that the statement is true for all $i < n-1$. It follows from [10, 1.13(4)] that $H^{n-1}_{I,J}(M) \cong H^{n-1}_{I,J}(M/\Gamma_{I,J}(M))$. Now let $\overline{M} = M/\Gamma_{I,J}(M)$ and $E = E(\overline{M})$ be an injective hull of $\overline{M}$. Note that E is an I-torsion-free R-module and $\mathrm{Supp}_R(H^{n-1}_{I,J}(M)) \subseteq \mathrm{Max}(R)$. The short exact sequence

$$0 \to \overline{M} \to E \to E/\overline{M} \to 0$$

gives that

$$H^{n-2}_{I,J}(E/\overline{M}) \cong H^{n-1}_{I,J}(\overline{M})$$

and

$$\mathrm{Ext}^i_R(R/I, E/\overline{M}) \cong \mathrm{Ext}^{i+1}_R(R/I, \overline{M})$$

for all $i \geq 0$. Since $\Gamma_{I,J}(M)$ is weakly artinian, by the assumption $\mathrm{Ext}^i_R(R/I, \overline{M})$ is weakly Laskerian for all $i < n$. Thus $\mathrm{Ext}^i_R(R/I, E/\overline{M})$ is weakly Laskerian for all $i < n-1$. Now by the inductive hypothesis $H^{n-2}_{I,J}(E/\overline{M})/JH^{n-2}_{I,J}(E/\overline{M})$ is weakly artinian and then the proof is complete. □

The following theorem establishes the relationship on the weakly artinianness between $H^i_I(M)$ and $H^i_{I,J}(M)$.

Theorem 3.7. *Let M be an R-module and t a non-negative integer. Assume that $H^i_{I,J}(M)$ is weakly artinian for all $i < t$. Then we have the following statements.*

(i) *$H^i_{\mathfrak{a}}(M)$ is weakly artinian for all $i < t$ and $\mathfrak{a} \in \tilde{W}(I, J)$.*
(ii) *$\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly artinian for all $i < t$ and $\mathfrak{a} \in \tilde{W}(I, J)$.*

Proof. (i) By [8, 10.47] there is a Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{a}}(H^q_{I,J}(M)) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{a}}(M).$$

Let $n < t$, there is a filtration Φ^i of $H^n = H^n_{\mathfrak{a}}(M)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \ldots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n_{\mathfrak{a}}(M)$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n/\Phi^{i+1}H^n.$$

Since $H^q_{I,J}(M)$ is weakly artinian for all $q < t$, we get $\dim H^q_{I,J}(M) = 0$ for all $q < t$ and then $E_2^{p,q} = 0$ for all $q < t$ and $p > 0$. Hence $E_\infty^{p,q} = 0$ for all $q < t$ and $p > 0$. This implies that

$$\Phi^1 H^n = \Phi^2 H^n = \ldots = \Phi^{n+1} H^n = 0$$

for all $n < t$. By the assumption, $E_2^{0,n} = \Gamma_{\mathfrak{a}}(H^n_{I,J}(M))$ is weakly artinian for all $n < t$. We consider the homomorphisms of the spectral sequence

$$0 = E_r^{-r,n+r-1} \to E_r^{0,n} \to E_r^{r,n-r+1} = 0$$

for all $r \geq 2$. It follows that

$$E_2^{0,n} = E_3^{0,n} = \ldots = E_\infty^{0,n}.$$

Now the short exact sequence

$$0 \to \Phi^1 H^n \to H^n_{\mathfrak{a}}(M) \to E_\infty^{0,n} \to 0$$

induces that $H^n_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(H^n_{I,J}(M))$ for all $n < t$, and the proof is complete.

(ii) By [8, 10.47] there is a Grothendieck spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_R(R/\mathfrak{a}, H^q_{I,J}(M)) \underset{p}{\Rightarrow} \operatorname{Ext}^{p+q}_R(R/\mathfrak{a}, M).$$

Let $n < t$, there are isomorphisms

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $i \leq n$ and $H^n = \operatorname{Ext}^n_R(R/\mathfrak{a}, M)$. By the hypothesis $E_2^{i,n-i}$ is weakly artinian for all $i \leq n$. Since $E_\infty^{i,n-i}$ is a subquotient of $E_2^{i,n-i}$, $E_\infty^{i,n-i}$ is weakly artinian. It should be mentioned that

$$E_\infty^{n-1,1} \cong \Phi^{n-1} H^n / \Phi^n H^n \cong \Phi^{n-1} H^n / E_\infty^{n,0}.$$

Then $\Phi^{n-1} H^n$ is weakly artinian. By descending induction, we conclude that $\Phi^{n-2} H^n, \ldots, \Phi^0 H^n$ are weakly artinian which completes the proof. □

Proposition 3.8. *Let M be a finitely generated $R-$module and t a non-negative integer. Assume that $H^i_{I,J}(R/\mathfrak{p})$ is weakly artinian for all $\mathfrak{p} \in \operatorname{Supp}(M)$ for all $i < t$. Then $H^i_{I,J}(M)$ is weakly artinian for all $i < t$.*

Proof. Since M is finitely generated, there is a filtration of submodules of M

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

such that $M_j/M_{j-1} \cong R/\mathfrak{p}_j$ for some $\mathfrak{p}_j \in \operatorname{Supp}(M)$. The short exact sequence

$$0 \to M_{j-1} \to M_j \to R/\mathfrak{p}_j \to 0$$

induces a long exact sequence

$$\cdots \to H^i_{I,J}(M_{j-1}) \to H^i_{I,J}(M_j) \to H^i_{I,J}(R/\mathfrak{p}_j) \to \cdots .$$

By induction on j, we conclude that $H^i_{I,J}(M)$ is weakly artinian for all $i < t$. □

The weakly filter regular sequence has been introduced in [5] as an extension of filter regular sequence. We will give a connection between the length of weakly filter regular sequence and weakly artinianness of $H^i_{I,J}(M)$.

Definition 3.9. ([5, 3.1]) Let $(R, \mathfrak{m})$ be a local ring, M an R-module and $x_1, \ldots, x_n$ a sequence of elements in $\mathfrak{m}$. We say that $x_1, \ldots, x_n$ is a weakly filter M-regular sequence if for each $i = 1, \ldots, n$, the module $0 :_{M/(x_1,\ldots,x_{i-1})M} x_i$ is weakly artinian.

Definition 3.10. ([5, 3.5]) The weakly filter depth of $\mathfrak{a}$ on M is defined as the length of any maximal weakly filter M-regular sequence in $\mathfrak{a}$, denoted by $w-f-\text{depth}(\mathfrak{a}, M)$.

Now we can characterize $w - f - \text{depth}(\mathfrak{a}, M)$ by the weakly artinianness of $H^i_{I,J}(M)$.

Theorem 3.11. *Let $(R, \mathfrak{m})$ be a local ring and M a weakly Laskerian R-module. Then*

$$\begin{aligned}\inf\{w - f - \textit{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \tilde{W}(I, J)\} &= \inf\{i \mid H^i_{I,J}(M) \textit{ is not weakly artinian}\}\\ &= \inf\{i \mid H^i_{I,J}(M) \ncong H^i_{\mathfrak{m},J}(M)\}\\ &= \inf\{i \mid H^i_{I,J}(M) \ncong H^i_{\mathfrak{m}}(M)\}.\end{aligned}$$

Proof. Let us prove the first equality. Set $n = \inf\{w-f-\text{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \tilde{W}(I, J)\}$. It follows from [4, 2.3] that $H^i_{\mathfrak{a}}(M)$ is weakly artinian for all $i < n$ and for all $\mathfrak{a} \in \tilde{W}(I, J)$. Note that $H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} H^i_{\mathfrak{a}}(M)$ by [10, 3.2]. We have by [9, 2.1] that

$$\text{Ass}(H^i_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a}\in\tilde{W}(I,J)} \text{Ass}(H^i_{\mathfrak{a}}(M)).$$

Since $\text{Ass}(H^i_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < n$ and $\mathfrak{a} \in \tilde{W}(I, J)$, we see that $\text{Ass}(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < n$. Thus $H^i_{I,J}(M)$ is weakly artinian for all $i < n$ and then $n \leq \inf\{i \mid H^i_{I,J}(M) \text{ is not weakly artinian}\}$.

Let $k = \inf\{i \mid H^i_{I,J}(M) \text{ is not weakly artinian}\}$, it remains to prove that $k \leq n$. Suppose, contrary to our claim, that $k > n$. Therefore $H^n_{I,J}(M)$ is weakly artinian. It follows from 3.7 (ii) that $\text{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly artinian for all $i \leq n$. By [5, 3.3], $\mathfrak{a}$ contains a weakly filter

M-regular sequence of length greater n for all $\mathfrak{a} \in \tilde{W}(I, J)$. Hence $n < \inf\{w - f - \operatorname{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \tilde{W}(I, J)\}$, a contradiction. Thus, we have the first equality.

The remain equalities follow from [6, 2.7], note that the module $H^i_{I,J}(M)$ is not weakly artinian if and only if $\operatorname{Supp}(H^i_{I,J}(M)) \not\subseteq \{\mathfrak{m}\}$. $\square$

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