New Hardy spaces of Musielak-Orlicz type and some related problems

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1. **New Hardy spaces of Musielak-Orlicz type**
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Holomorphic Hardy spaces $\mathcal{H}^p(\mathbb{D})$

Definition 1.1 (Hardy[1] and Riesz[2])

Let $0 < p < \infty$. The Hardy space $\mathcal{H}^p(\mathbb{D})$ is defined as the space of holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{H}^p(\mathbb{D})} := \sup_{0<r<1} \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right]^{1/p} < \infty.$$

When $p = \infty$, the Hardy space $\mathcal{H}^\infty(\mathbb{D})$ is then defined as the space of bounded holomorphic functions f on $\mathbb{D}$ with

$$\|f\|_{\mathcal{H}^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}} |f(z)|.$$

[1] G. H. Hardy, The mean value of the modulus of an analytic function. Proc. London Math. Soc., 14(2) (1915), 269-277.

[2] F. Riesz, Über die Randwerte einer analytischen Funktion. Math. Z. 18 (1923), no. 1, 87-95.

Holomorphic Hardy spaces $\mathcal{H}^p(\mathbb{C}_+)$

Definition 1.2 (Stein-Weiss[3])

Let $0 < p < \infty$. The Hardy space $\mathcal{H}^p(\mathbb{C}_+)$ is defined as the space of holomorphic functions f on $\mathbb{C}_+$ such that

$$\|f\|_{\mathcal{H}^p(\mathbb{C}_+)} := \sup_{y>0} \left[\int_{-\infty}^{\infty} |f(x+iy)|^p dx\right]^{1/p} < \infty.$$

When $p = \infty$, the Hardy space $\mathcal{H}^\infty(\mathbb{C}_+)$ is then defined as the space of bounded holomorphic functions f on $\mathbb{C}_+$ with

$$\|f\|_{\mathcal{H}^\infty(\mathbb{C}_+)} := \sup_{z\in\mathbb{C}_+} |f(z)|.$$

[3] E. M. Stein and G. Weiss, On the theory of harmonic functions of several variables, I: The theory of H^p spaces. Acta Math., 103 (1960), 26-62.

The dual spaces of H^p

Theorem 1.3

Let $1 < p < \infty$ and $1/q + 1/p = 1$. Then, the dual space of $\mathcal{H}^p(\mathbb{C}_+)$ is $\mathcal{H}^q(\mathbb{C}_+)$.

A natural question is that:

Question 1.1

$(\mathcal{H}^1(\mathbb{C}_+))^* = \mathcal{H}^\infty(\mathbb{C}_+)$?

The real Hardy space $H^1(\mathbb{R})$

Denote by $P(t) = \frac{1}{1+t^2}$, $t \in \mathbb{R}$, the Poisson kernel on $\mathbb{R}$ and $P_y(t) = \frac{1}{y} P(t/y)$, $y > 0$. The real Hardy space $H^1(\mathbb{R})$ is then defined as the set of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_{H^1(\mathbb{R})} := \|\mathcal{M}f\|_{L^1} < \infty,$$

where $\mathcal{M}f(x) := \sup_{|t-x|<y} |P_y * f(t)|$ for every $x \in \mathbb{R}$.

Theorem 1.4 (Burkholder-Gundy-Silverstein[4])

$F(z) \in \mathcal{H}^1(\mathbb{C}_+)$ iff there exists a function $f(t) \in H^1(\mathbb{R})$ such that

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{z-t} dt.$$

[4] D. L. Burkholder, R. F. Gundy and M. L. Silverstein, A maximal function characterization of the class H^p. Trans. Amer. Math. Soc. 157 (1971), 137-153.

The dual space of $H^1(\mathbb{R})$

Definition 1.5 (John-Nirenberg[5])

A function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is said to be in the space $BMO(\mathbb{R})$ if

$$\|f\|_{BMO(\mathbb{R})} := \sup_B \frac{1}{|B|} \int_B \left| f(x) - \frac{1}{|B|} \int_B f(y)dy \right| dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}$.

Theorem 1.6 (Fefferman-Stein[6])

The space $BMO(\mathbb{R})$ is just the dual space $H^1(\mathbb{R})$.

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[5] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math., 14 (1961), 415-426.

[6] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math., 129 (1972), 137-193.

The dual space of $H^1(\mathbb{R})$

But this duality is not like the duality $L^p - L^q$. More precisely, product of a function $f \in H^1(\mathbb{R})$ and a function $g \in BMO(\mathbb{R})$ need not be integrable. A natural question is that:

Question 1.2

What can one say about such products?

On the product of functions in $H^1(\mathbb{R})$ and $BMO(\mathbb{R})$

For a function $f \in H^1(\mathbb{R})$ and a function $g \in BMO(\mathbb{R})$, then the product $f \times g$ can be defined as follows

$$\langle f \times g, \phi \rangle = \langle \phi g, f \rangle_{BMO-H^1}, \quad \phi \in \mathcal{S}(\mathbb{R}).$$

Let $H^{\Phi}_w(\mathbb{R})$ be the weighted Hardy-Orlicz space associated with the Orlicz function $\Phi(t) = t/\log(e+t)$ and the Muckenhoupt weight $w(x) = 1/\log(e+|x|)$.

On the product of functions in $H^1(\mathbb{R})$ and $BMO(\mathbb{R})$

Theorem 1.7 (Bonami-Iwaniec-Jones-Zinsmeister[7])

Let $f \in H^1(\mathbb{R})$. Then, there exist two bounded linear operators $\mathcal{L}_f : BMO(\mathbb{R}) \to L^1(\mathbb{R})$ and $\mathcal{H}_f : BMO(\mathbb{R}) \to H^\Phi_w(\mathbb{R})$ such that

$$f \times g = \mathcal{L}_f(g) + \mathcal{H}_f(g), \quad \forall g \in BMO(\mathbb{R}).$$

Conjecture 1.1

There exist two bounded linear operators
$\mathcal{L} : H^1(\mathbb{R}) \times BMO(\mathbb{R}) \to L^1(\mathbb{R})$ and
$\mathcal{H} : H^1(\mathbb{R}) \times BMO(\mathbb{R}) \to H^\Phi_w(\mathbb{R})$ such that

$$f \times g = \mathcal{L}(f,g) + \mathcal{H}(f,g), \quad \forall (f,g) \in H^1(\mathbb{R}) \times BMO(\mathbb{R}).$$

[7] A. Bonami, T. Iwaniec, P. Jones, P Zinsmeister, Michel On the product of functions in *BMO* and H^1. Ann. Inst. Fourier (Grenoble) 57 (2007), no. 5, 1405–1439.

On the product of functions in $H^1(\mathbb{R})$ and $BMO(\mathbb{R})$

Theorem 1.8 (Bonami-Grellier-K.[8])

There exist two bounded linear operators $\mathcal{L} : H^1(\mathbb{R}) \times BMO(\mathbb{R}) \to L^1(\mathbb{R})$ *and* $\mathcal{H} : H^1(\mathbb{R}) \times BMO(\mathbb{R}) \to H^{\log}(\mathbb{R})$ *such that*

$$f \times g = \mathcal{L}(f, g) + \mathcal{H}(f, g), \quad \forall (f, g) \in H^1(\mathbb{R}) \times BMO(\mathbb{R}).$$

Here the space $H^{\log}(\mathbb{R})$ is a new Hardy space of Musielak-Orlicz type associated with the Musielak-Orlicz function $\varphi(x, t) = \frac{t}{\log(e+t)+\log(e+|x|)}$.

[8] A. Bonami, S. Grellier and L. D. Ky, Paraproducts and products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ through wavelets. J. Math. Pures Appl. (9) 97 (2012), no. 3, 230–241.

[9] A. Bonami, J. Cao, L. D. Ky, L. Liu, D. Yang and W. Yuan, Multiplication between Hardy spaces and their dual spaces. J. Math. Pures Appl. (9) 131 (2019), 130–170.

New Hardy spaces of Musielak-Orlicz type

$\varphi : \mathbb{R} \times [0, \infty)$ is said to be a Musielak-Orlicz function if

1. $\varphi(x, \cdot)$ is an Orlicz function satisfying

$$\varphi(x, st) \leq Cs^p \varphi(x, t), \quad \forall x \in \mathbb{R}, s \in (0, 1], t \in [0, \infty),$$

where $0 < p \leq 1$, and

$$\varphi(x, st) \leq Cs\varphi(x, t), \quad \forall x \in \mathbb{R}, s \in [1, \infty), t \in [0, \infty).$$

2. $\varphi(\cdot, t)$ is a Muckenhoupt weight in A_q $(1 < q < \infty)$, that is

$$\frac{1}{|B|} \int_B \varphi(x, t) dx \left(\frac{1}{|B|} (\varphi(x, t))^{-\frac{1}{q-1}} dx \right)^{q-1} \leq C$$

for all $t \geq 0$ and interval $B \subset \mathbb{R}$.

New Hardy spaces of Musielak-Orlicz type

Definition 1.9 (K.[10])

Let $\varphi : \mathbb{R} \times [0, \infty)$ be a Musielak-Orlicz function. The Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R})$ is then defined as the completion of $\{f \in L^2(\mathbb{R}) : \mathcal{M}f \in L^\varphi(\mathbb{R})\}$ in the norm

$$\|f\|_{H^\varphi(\mathbb{R})} := \|\mathcal{M}f\|_{L^\varphi(\mathbb{R})} < \infty.$$

[10] L. D. Ky, New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators. Integral Equations Operator Theory 78 (2014), no. 1, 115–150.

New Hardy spaces of Musielak-Orlicz type

A bounded function a is a φ-atom if
i) supp $a \subset B$ for some ball B,
ii) $\|a\|_{L^\infty} \leq \|\chi_B\|_{L^\varphi}^{-1}$,
iii) $\int_{\mathbb{R}} a(x)x^\alpha dx = 0$ for any $|\alpha| \leq m(\varphi)$.
We next define the *atomic Hardy space of Musielak-Orlicz type* $H^{\varphi(\cdot,\cdot)}_{\rm at}(\mathbb{R})$ as those distributions $f \in \mathcal{S}'(\mathbb{R})$ such that $f = \sum_j b_j$ (in the sense of $\mathcal{S}'(\mathbb{R})$), where b_j's are multiples of φ-atoms supported in the balls B_j's, with the property $\sum_j \varphi(B_j, \|b_j\|_{L^\infty}) < \infty$; and define the norm of f by

$$\|f\|_{H^{\varphi(\cdot,\cdot)}_{\rm at}} = \inf\left\{\Lambda_\infty(\{b_j\}) : f = \sum_j b_j \quad \text{in the sense of } \mathcal{S}'(\mathbb{R})\right\},$$

where $\Lambda_\infty(\{b_j\}) = \inf\left\{\lambda > 0 : \sum_j \varphi\left(B_j, \frac{\|b_j\|_{L^\infty}}{\lambda}\right) \leq 1\right\}$ with $\varphi(B,t) := \int_B \varphi(x,t)dx$ for all $t \geq 0$ and B is measurable.

New Hardy spaces of Musielak-Orlicz type

Theorem 1.10 (K.[10])

$H_{\mathrm{at}}^{\varphi(\cdot,\cdot)}(\mathbb{R}) = H^{\varphi(\cdot,\cdot)}(\mathbb{R})$ *with equivalent norms.*

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[10] L. D. Ky, New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators. Integral Equations Operator Theory 78 (2014), no. 1, 115–150.

[11] D. Yang, Y. Liang and L. D. Ky, Real-variable theory of Musielak-Orlicz Hardy spaces. Lecture Notes in Mathematics, 2182. Springer, Cham, 2017.

Unweighted estimates for commutators of SIOs

Let $\delta \in (0,1]$. A bounded linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is said to be a δ-Calderón-Zygmund singular integral operator (SIO) if T can be written as

$$Tf(x) = \int_{\mathbb{R}} K(x,y) f(y)\,dy$$

for all $f \in C_c(\mathbb{R})$ and $x \notin \operatorname{supp} f$, where $K : \mathbb{R} \times \mathbb{R} \setminus \{(x,x) : x \in \mathbb{R}\} \to \mathbb{C}$ satisfies $|K(x,y)| \leq \frac{C}{|x-y|}$ and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C \frac{|x-x'|^{\delta}}{|x-y|^{1+\delta}}$$

for all $2|x-x'| \leq |x-y|$.

[12] A. P. Calderón and A. Zygmund, On the existence of certain singular integrals. Acta Math. 88 (1952), 85–139.

Unweighted estimates for commutators of SIOs

Let $b \in L^1_{\mathrm{loc}}(\mathbb{R})$, we define the commutator

$$[b, T](f) := bTf - T(bf).$$

Theorem 2.1 (Coifman-Rochberg-Weiss[13])

Let $b \in BMO(\mathbb{R})$. Then, for every $p \in (1, \infty)$, the commutator $[b, T]$ is bounded on $L^p(\mathbb{R})$.

[13] R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables. Ann. of Math. (2) 103 (1976), no. 3, 611–635.

Unweighted estimates for commutators of SIOs

A bounded function a is a H^1_b-atom if
i) supp $a \subset B$ for some ball B,
ii) $\|a\|_{L^\infty} \leq |B|^{-1}$,
iii) $\int_{\mathbb{R}} a(x)dx = \int_{\mathbb{R}} a(x)b(x)dx = 0$.
Following Pérez[12], we define the Hardy space $H^1_b(\mathbb{R})$ as those functions $f \in L^1(\mathbb{R})$ such that $f = \sum_j \lambda_j a_j$, where a_j's are H^1_b-atoms and $\sum_{j=1}^\infty |\lambda_j| < \infty$. The norm on $H^1_b(\mathbb{R})$ is then defined by

$$\|f\|_{H^1_b} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_j \lambda_j a_j \right\}.$$

Unweighted estimates for commutators of SIOs

Theorem 2.2 (Pérez[14])

Let $b \in BMO(\mathbb{R})$. Then, $[b, T]$ is bounded from $H^1_b(\mathbb{R})$ into $L^1(\mathbb{R})$.

We define $\mathcal{H}^1_b(\mathbb{R})$ as the set of functions $f \in H^1(\mathbb{R})$ such that

$$\|f\|_{\mathcal{H}^1_b} := \|f\|_{H^1}\|b\|_{BMO} + \|[b, H](f)\|_{L^1} < \infty.$$

Theorem 2.3 (K.[15])

Let $b \in BMO(\mathbb{R})$. Then, $[b, T]$ is bounded from $\mathcal{H}^1_b(\mathbb{R})$ into $L^1(\mathbb{R})$ for every Calderón-Zygmund singular integral operator T. Furthermore, $\mathcal{H}^1_b(\mathbb{R})$ is the largest space having this property.

[14] C. Pérez, Endpoint estimates for commutators of singular integral operators. J. Funct. Anal. 128 (1995), 163–185.

[15] L. D. Ky, Bilinear decompositions and commutators of singular integral operators. Trans. Amer. Math. Soc. 365 (2013), no. 6, 2931–2958.

Weighted estimates for commutators of SIOs

Let $w \in A_\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \frac{w(x)}{1+|x|}\,dx < \infty$. A locally integrable function b is said to be in $\mathcal{BMO}_w(\mathbb{R})$ if

$$\|b\|_{\mathcal{BMO}_w} := \sup_{B\subset\mathbb{R}} \left\{ \frac{1}{w(B)} \left[\int_{B^c} \frac{w(x)}{|x-x_B|}\,dx\right] \left[\int_B |b(y)-b_B|\,dy\right] \right\} < \infty$$

Theorem 2.4 (Liang-K.-Yang[16])

Let $\delta \in (0,1]$, $w \in A_{1+\delta}(\mathbb{R})$ satisfy $\int_{\mathbb{R}} \frac{w(x)}{1+|x|}\,dx < \infty$ and $b \in \mathcal{BMO}_w(\mathbb{R})$. Then,for every δ-Calderón-Zygmund operator T, the commutator $[b, T]$ is bounded from $H^1_w(\mathbb{R})$ into $L^1_w(\mathbb{R})$

[16] Y. Liang, L. D. Ky and D. Yang, Weighted endpoint estimates for commutators of Calderón-Zygmund operators. Proc. Amer. Math. Soc. 144 (2016), no. 12, 5171–5181.

Thank you very much for your attention!