Title: Regularity theory for second order partial differential equation arising from composite material

Authors: Nguyen Huyen-Thu Ho, Bach Ngoc-My Nguyen, Quoc-Hung Nguyen, Vu Tien-Anh Nguyen, Ngoc-Huy Pham, Minh-Nguyen Tran, Quang-Minh Tran and Tuan-Dat Tran.

Abstract: We establish a new theory of regularity for second order partial differential equation arising from composite material:

$$\begin{cases} -\operatorname{div}(A\nabla u) + \lambda u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{0.1}$$

where $\lambda \geq 0$;
(i) Ω, Ω_1 are smooth bounded simply connected open subsets of $\mathbb{R}^d$ with $d \geq 2$ satisfying $\Omega_1 \subset\subset \Omega$, and $\Omega_2 = \Omega \backslash \overline{\Omega}_1$;
(ii) A is given by

$$A(x) = \begin{cases} A_1(x) & \text{in } \Omega_1, \\ -A_2(x) & \text{in } \Omega_2; \end{cases}$$

A_1, A_2 are two real, symmetric matrix-valued functions in Ω_1 and Ω_2 respectively satisfying uniformly elliptic i.e for some constant $\Lambda \geq 1$, one has, for $j = 1, 2$,

$$\Lambda^{-1}|\xi|^2 \leq \langle A_j(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d, \text{ for a.e. } x \in \Omega_j. \tag{0.2}$$

Our work is stimulated by the study of composite media with closely spaced interfacial boundaries see [5]. Since A is not elliptic, a priori estimates of (0.1) are non-standard and do not hold in general. In this paper, we develop a new method to show existence and regularity of solutions to (0.1). It is based on the Fourier analysis and Calderón-Zygmund theory. Namely, we prove that

Theorem 1. *If $A_1, A_2 \in C^1(\overline{\Omega})$ satisfy the following conditions, with respect to $e = \nu(x)$, $x \in \partial\Omega_1$*

$$\langle A_2(x)e, e\rangle\langle A_2(x)\xi, \xi\rangle - \langle A_2(x)e, \xi\rangle^2 \neq \langle A_1(x)e, e\rangle\langle A_1(x)\xi, \xi\rangle - \langle A_1(x)e, \xi\rangle^2$$

and

$$\langle A_2(x)e, e\rangle \neq \langle A_1(x)e, e\rangle$$

for any $\xi \in \mathcal{P}(x) \setminus \{0\}$ where $\nu(x)$ denotes the outward normal unit vector on $\partial\Omega_1$ at x,

$$\mathcal{P}(x) := \{\xi \in \mathbb{R}^d; \langle \xi, e\rangle = 0\}.$$

There exists $\lambda_0 > 0$ such that for any $f \in L^p(\Omega)$, $1 < p < \infty$ and $\lambda \geq \lambda_0$, then the equation (0.1) admits a unique solution u in $W^{2,p}(\Omega_1 \cup \Omega_2) \cap W_0^{1,p}(\Omega)$ satisfying

$$||D^2u||_{L^p(\Omega_1\cup\Omega_2)} + \lambda||u||_{L^p} \lesssim ||f||_{L^p(\Omega)}. \tag{0.3}$$

In particular, if $p > \frac{d}{2}$,

$$||\nabla u||_{L^\infty(\Omega)} \lesssim \lambda^{\frac{d}{2p}-\frac{1}{2}} ||f||_{L^p(\Omega)}. \tag{0.4}$$

Moreover, if $A_1, A_2 \in C^\alpha$ *for* $\alpha \in (0,1)$*, then*

$$||D^2 u||_{C^\alpha(\Omega_1 \cup \Omega_2)} + \lambda ||u||_{C^\alpha(\Omega)} \lesssim ||f||_{C^\alpha(\Omega)}. \tag{0.5}$$

Note that (0.3) and (0.5) is well known when A is uniformly elliptic, see [1, 3, 4, 5]. We also show that the conditions on A_1, A_2 in Theorem 1 are necessary and sufficient for estimates (0.3) and (0.5) .

Natural and interesting questions on the composite material/inverse scattering theory include discreteness of the spectrum, the Weyl law of eigenvalues and the completeness of the eigenfunctions to the following equation: for $\lambda \in \mathbb{R}$,

$$\begin{cases} -\operatorname{div}(A\nabla u) = \lambda u \ \text{ in } \Omega, \\ \quad u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{0.6}$$

We show that if the assumptions of theorem 1 holds then the spectrum of (0.6) is discrete, the completeness of the generalized eigenfunctions and the Weyl law for transmission eigenvalues of (0.6) holds. More precisely, we have

Theorem 2. *Assume the assumptions of 1 holds. Then, the spectrum of the problem* (0.6) *is discrete and the generalized eigenfunctions of* (0.6) *are complete in* $L^2(\Omega)$*. Moreover, for eigenvalues* $\{\lambda_n\}_n$ *of* (0.6) *with* $\lambda_1 \le \lambda_2 \le ... \le \lambda_n \le ...$*, there holds*

$$N(t) := \#\Big\{k \in \mathbb{N} : |\lambda_k| \le t\Big\} = \mathbf{c} t^{\frac{d}{2}} + o(t^{\frac{d}{2}}) \ \textit{as} \ \ t \to +\infty, \tag{0.7}$$

where

$$\mathbf{c} = \frac{1}{(2\pi)^d} \sum_{j=1}^{2} \int_{\Omega_j} \big|\{\xi \in \mathbb{R}^d : \langle A_j(x)\xi, \xi\rangle < \Sigma_j(x)\}\big| \, dx. \tag{0.8}$$

Here for a measurable subset D *of* $\mathbb{R}^d$*, we denote* $|D|$ *its (Lebesgue) measure.*

The proof of theorem (2) is new and based on the L^p regularity (0.3) and a subtle application of the spectral theory for the Hilbert Schmidt operators in [2].

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Nguyen Huyen-Thu Ho
Email: hnht1211.hcmus@gmail.com
Faculty of Mathematics and Computer Science, University of Science, VNU-HCM, 227 Nguyen Van Cu street, District 5,Ho Chi Minh City, Vietnam

Bach Ngoc-My Nguyen
Email: NgocMyNguyenBach@gmail.com
Faculty of Mathematics and Computer Science University of Science, VNU-HCM, 227 Nguyen Van Cu street, District 5, Ho Chi Minh City, Vietnam

Quoc-Hung Nguyen
Email: qhnguyen@shanghaitech.edu.cn
ShanghaiTech University, 393 Middle Huaxia Road, Pudong, Shanghai, 201210, China

Vu Tien-Anh Nguyen
Email: nguyenvutienanh1@gmail.com
Faculty of Mathematics and Computer Science University of Science, VNU-HCM, 227 Nguyen Van Cu street, District 5, Ho Chi Minh City, Vietnam

Ngoc-Huy Pham
Email: cqtk11phamngochuy@gmail.com
Faculty of Mathematics - Informatics, Ho Chi Minh City University of Education, 280 An Duong Vuong street, District 5, Ho Chi Minh City, Vietnam

Minh-Nguyen Tran
Email: minhnguyent1110@gmail.com
Faculty of Mathematics and Computer Science University of Science, VNU-HCM, 227 Nguyen Van Cu street, District 5, Ho Chi Minh City, Vietnam

Quang-Minh Tran (corresponding author)
Email: tqminh.khtn93@gmail.com
Faculty of Mathematics and Computer Science University of Science, VNU-HCM, 227 Nguyen Van Cu street, District 5, Ho Chi Minh City, Vietnam

Tuan-Dat Tran

Email: tuandat.maths@gmail.com
Faculty of Mathematics and Computer Science University of Science, VNU-HCM, 227 Nguyen Van Cu street, District 5, Ho Chi Minh City, Vietnam