A note on the Gagliardo-Nirenberg interpolation inequalities using Lorentz, Sobolev, and BMO spaces

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Abstract. In this paper, we would like to study the Gagliardo-Nirenberg interpolation inequalities using Lorentz, Sobolev, and BMO spaces. The classical inequalities of this type are as follows:

$$\|f\|_{L^p} \le C\|f\|^{\theta}_{L^q}\|f\|^{1-\theta}_{\dot{H}^s}, \tag{0.1}$$

with

$$\frac{1}{p} = \frac{\theta}{q} + (1-\theta)(\frac{1}{2} - \frac{s}{n}).$$

After that, Mc. Cormick et al. improved (0.1) by showing that

Theorem 0.1 (McCormick et al.) *Let $1 \le q < p$, and $s \ge 0$ with $s > n(1/2 - 1/p)$. There is a constant $C = C(p,q,s)$ such that if $f \in L^{q,\infty}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$ then $f \in L^p(\mathbb{R}^n)$ and*

$$\|f\|_{L^p} \le C\|f\|^{\theta}_{L^{q,\infty}}\|f\|^{1-\theta}_{\dot{H}^s}. \tag{0.2}$$

In this note, we want to enhance inequality (0.2) by replacing the term $\|f\|_{L^p}$ by $\|f\|_{L^{p,\alpha}}$, for any $\alpha > 0$ (note that $\|f\|_{L^{p,\alpha}}$ is a Lorentz space). Moreover, we also prove an interpolation inequality by the means of Lorentz spaces and BMO spaces. Our study is inspired by [1, 2, 3] and their references.

References

[1] David S. Mc Cormick, James C. Robinson, and Jose L. Rodrigo, *Generalised GagliardoNirenberg Inequalities Using Weak Lebesgue Spaces and BMO*, Milan J. Math. **81** (2013) 265-289.

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